Hopf cyclic cohomology, Hodge theory, Proper actions (joint with Xiang Tang and Weiping Zhang)

姚一隽

复旦大学

华东师范大学, 2011年6月24日

1. Cyclic cohomology of Hopf algebroids
   - Hopf algebroids
   - Hopf algebroid $\mathcal{H}(G, M)$
   - Cyclic Cohomology
   - Hopf cyclic cohomology of $\mathcal{H}(G, M)$

2. Generalized Hodge theory

Basic Setting

Basic Setting

- G a Lie group, acting on a smooth manifold M.

Basic Setting

- G a Lie group, acting on a smooth manifold M.
- the action is cocompact and proper.

A, B : unital topological algebras.

A, B : unital topological algebras. A (topological) bialgebroid structure on A, over B, is

A, B : unital topological algebras. A (topological) bialgebroid structure on A, over B, is

- *source map* $\alpha : B \to A$, *target map*: $\beta : B \to A$, satisfying $\alpha(a)\beta(b) = \beta(b)\alpha(a)$, for all $a, b \in A$.

A, B : unital topological algebras. A (topological) bialgebroid structure on A, over B, is

- *source map* $\alpha : B \to A$, *target map*: $\beta : B \to A$, satisfying $\alpha(a)\beta(b) = \beta(b)\alpha(a)$, for all $a, b \in A$.
- *coproduct*: continuous B-B bimodule map $\Delta : A \to A \otimes_B A$,
  1. $\Delta(1) = 1 \otimes 1$;
  2. $(\Delta \otimes_B Id)\Delta = (Id \otimes_B \Delta)\Delta : A \to A \otimes_B A \otimes_B A$,
  3. $\Delta(a)(\beta(b) \otimes 1 - 1 \otimes \alpha(b)) = 0$, for $a \in A$, $b \in B$,
  4. $\Delta(a_1 a_2) = \Delta(a_1)\Delta(a_2)$, for $a_1, a_2 \in A$.

A, B : unital topological algebras. A (topological) bialgebroid structure on A, over B, is

- *source map* $\alpha : B \to A$, *target map*: $\beta : B \to A$, satisfying $\alpha(a)\beta(b) = \beta(b)\alpha(a)$, for all $a, b \in A$.
- *coproduct*: continuous B-B bimodule map $\Delta : A \to A \otimes_B A$,
  1. $\Delta(1) = 1 \otimes 1$;
  2. $(\Delta \otimes_B Id)\Delta = (Id \otimes_B \Delta)\Delta : A \to A \otimes_B A \otimes_B A$,
  3. $\Delta(a)(\beta(b) \otimes 1 - 1 \otimes \alpha(b)) = 0$, for $a \in A$, $b \in B$,
  4. $\Delta(a_1 a_2) = \Delta(a_1)\Delta(a_2)$, for $a_1, a_2 \in A$.
- *counit*: continuous B-B bimodule map $\epsilon : A \to B$,
  1. $\epsilon(1) = 1$;
  2. $\ker \epsilon$ is a left A ideal;
  3. $(\epsilon \otimes_B Id)\Delta = (Id \otimes \epsilon)\Delta = Id : A \to A$
  4. For any $a, a' \in A$, $b, b' \in B$, $\epsilon(\alpha(b)\beta(b')a) = b\epsilon(a)b'$, and $\epsilon(aa') = \epsilon(a\alpha(\epsilon(a'))) = \epsilon(a\beta(\epsilon(a')))$.

A topological *para Hopf algebroid*

A topological *para Hopf algebroid* = bialgebroid A, over B, A topological *para Hopf algebroid* = bialgebroid A, over B, + a continuous algebra anti-isomorphism $S : A \to A$ such that

$$S^2 = Id, \quad S\beta = \alpha, \quad m_A(S \otimes_B Id)\Delta = \beta\epsilon S : A \to A,$$

and

$$S(a^{(1)})^{(1)} a^{(2)} \otimes_B S(a^{(1)})^{(2)} = 1 \otimes_B S(a).$$

A topological *para Hopf algebroid* = bialgebroid A, over B, + a continuous algebra anti-isomorphism $S : A \to A$ such that

$$S^2 = Id, \qquad S\beta = \alpha, \qquad m_A(S \otimes_B Id)\Delta = \beta\epsilon S : A \to A,$$

and

$$S(a^{(1)})^{(1)}a^{(2)} \otimes_B S(a^{(1)})^{(2)} = 1 \otimes_B S(a).$$

Sweedler's notation for the coproduct $\Delta(a) = a^{(1)} \otimes_B a^{(2)}$.

- B: algebra of differential forms on M,

- B: algebra of differential forms on M,
- A : algebra of B-valued functions on G.

- B: algebra of differential forms on M,
- A : algebra of B-valued functions on G.
- Both A and B are differential graded algebras with the de Rham differential.

- B: algebra of differential forms on M,
- A : algebra of B-valued functions on G.
- Both A and B are differential graded algebras with the de Rham differential.
- $g^*(a)(x) := a(gx)$.

$$\alpha(b)(g) = b, \qquad \text{and} \qquad \beta(b)(g) = g^*(b).$$

- B: algebra of differential forms on M,
- A : algebra of B-valued functions on G.
- Both A and B are differential graded algebras with the de Rham differential.
- $g^*(a)(x) := a(gx)$.

$$\alpha(b)(g) = b, \qquad \text{and} \qquad \beta(b)(g) = g^*(b).$$

- $A \otimes_B A$ is isomorphic to the space of B-valued functions on $G \times G$, i.e.

$$(\phi \otimes_B \psi)(g_1, g_2) = \phi(g_1)g_1^*(\psi(g_2))$$

for $\phi, \psi \in A$.

- B: algebra of differential forms on M,
- A : algebra of B-valued functions on G.
- Both A and B are differential graded algebras with the de Rham differential.
- $g^*(a)(x) := a(gx)$.

$$\alpha(b)(g) = b, \qquad \text{and} \qquad \beta(b)(g) = g^*(b).$$

- $A \otimes_B A$ is isomorphic to the space of B-valued functions on $G \times G$, i.e.

$$(\phi \otimes_B \psi)(g_1, g_2) = \phi(g_1)g_1^*(\psi(g_2))$$

for $\phi, \psi \in A$.
- Define $\Delta : A \to A \otimes_B A$ by

$$\Delta(\phi)(g_1, g_2) = \phi(g_1 g_2),$$

- B: algebra of differential forms on M,
- A : algebra of B-valued functions on G.
- Both A and B are differential graded algebras with the de Rham differential.
- $g^*(a)(x) := a(gx)$.

$$\alpha(b)(g) = b, \qquad \text{and} \qquad \beta(b)(g) = g^*(b).$$

- $A \otimes_B A$ is isomorphic to the space of B-valued functions on $G \times G$, i.e.

$$(\phi \otimes_B \psi)(g_1, g_2) = \phi(g_1)g_1^*(\psi(g_2))$$

for $\phi, \psi \in A$.
- Define $\Delta : A \to A \otimes_B A$ by

$$\Delta(\phi)(g_1, g_2) = \phi(g_1 g_2),$$

- counit map $\epsilon : A \to B$, $\epsilon(\phi) = \psi(1)$, for $\phi \in A$.

we define the antipode on A by

$$S(\phi)(g) = g^*(\phi(g^{-1})).$$

It is easy to check that S satisfies properties for an antipode of a para Hopf algebroid. We denote this Hopf algebroid by $\mathcal{H}(G, M)$.

cyclic module $A^\natural$ for $(A, B, \alpha, \beta, \Delta, \epsilon, S)$

cyclic module $A^\natural$ for $(A, B, \alpha, \beta, \Delta, \epsilon, S)$

- Define

$$C^0 = B, \qquad C^n = \underbrace{A \otimes_B A \otimes_B \cdots \otimes_B A}_{n}, \; n \geq 1.$$

cyclic module $A^\natural$ for $(A, B, \alpha, \beta, \Delta, \epsilon, S)$

- Define

$$C^0 = B, \qquad C^n = \underbrace{A \otimes_B A \otimes_B \cdots \otimes_B A}_{n}, \; n \geq 1.$$

- Faces and degeneracy operators are defined as follows:

$$\delta_0(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = 1 \otimes_B a^1 \otimes_B \cdots \otimes_B a^{n-1};$$
$$\delta_i(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B \Delta a^i \otimes_B \cdots \otimes_B a^{n-1}, \; 1 \leq i \leq n-1;$$
$$\delta_n(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B a^{n-1} \otimes_B 1;$$
$$\sigma_i(a^1 \otimes_B \cdots \otimes_B a^{n+1}) = a^1 \otimes_B \cdots \otimes_B a^i \otimes_B \epsilon(a^{i+1}) \otimes_B a^{i+2} \cdots \otimes_B a^{n+1}.$$

cyclic module $A^\natural$ for $(A, B, \alpha, \beta, \Delta, \epsilon, S)$

- Define

$$C^0 = B, \qquad C^n = \underbrace{A \otimes_B A \otimes_B \cdots \otimes_B A}_{n},\ n \geq 1.$$

- Faces and degeneracy operators are defined as follows:

$$\delta_0(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = 1 \otimes_B a^1 \otimes_B \cdots \otimes_B a^{n-1};$$
$$\delta_i(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B \Delta a^i \otimes_B \cdots \otimes_B a^{n-1},\ 1 \leq i \leq n-1;$$
$$\delta_n(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B a^{n-1} \otimes_B 1;$$
$$\sigma_i(a^1 \otimes_B \cdots \otimes_B a^{n+1}) = a^1 \otimes_B \cdots \otimes_B a^i \otimes_B \epsilon(a^{i+1}) \otimes_B a^{i+2} \cdots \otimes_B a^{n+1}.$$

- The cyclic operators are given by

$$\tau_n(a^1 \otimes_B \cdots \otimes_B a^n) = (\Delta^{n-1} S(a^1))(a^2 \otimes \cdots a^n \otimes 1).$$

Theorem (Kaminker-Tang)

Let G be a Lie group acting on a smooth manifold.

$$HC^{\bullet}(\mathcal{H}(G, M)) = \oplus_{k\geq 0} H^{\bullet - 2k}(G; (\Omega^*(M), d)).$$

- Let $\Omega^*(M)^G$ be the space of G-invariant differential forms on M, which inherits a natural de Rham differential d. If G acts on M properly, then the differentiable cohomology $H^\bullet(G; (\Omega^*(M), d))$ is computed as follows,

$$H^\bullet(G; (\Omega^*(M), d)) = H^\bullet(\Omega^*(M)^G, d).$$

- Let $\Omega^*(M)^G$ be the space of G-invariant differential forms on M, which inherits a natural de Rham differential d. If G acts on M properly, then the differentiable cohomology $H^\bullet(G; (\Omega^*(M), d))$ is computed as follows,

$$H^\bullet(G; (\Omega^*(M), d)) = H^\bullet(\Omega^*(M)^G, d).$$

Proposition

If G acts on M properly, then we have

$$HC^\bullet(\mathcal{H}(G, M)) = \oplus_{k\geq 0} H^{\bullet-2k}(\Omega^*(M)^G, d)$$

and

$$HP^\bullet(\mathcal{H}(G, M)) = \oplus_{k\in\mathbb{Z}} H^{\bullet+2k}(\Omega^*(M)^G, d).$$

Theorem (Tang-Y-Zhang)

The cyclic cohomology groups of $\mathcal{H}(G, M)$ are of finite dimension.

Theorem (Tang-Y-Zhang)

The cyclic cohomology groups of $\mathcal{H}(G, M)$ are of finite dimension.

all we need to prove is:

The cohomology groups $H^\bullet(G; (\Omega^*(M), d))$ are of finite dimension.

- G acts on M properly $\Rightarrow$ G-invariant metric on M.

- G acts on M properly$\Rightarrow$ G-invariant metric on M.
- M/G is compact$\Rightarrow$ compact $Y \subset M$ such that $G(Y) = M$.

- G acts on M properly$\Rightarrow$ G-invariant metric on M.
- M/G is compact$\Rightarrow$ compact $Y \subset M$ such that $G(Y) = M$.
- inner product

$$\langle \alpha, \beta \rangle_0 = \int_M \alpha \wedge *\beta.$$

- G acts on M properly$\Rightarrow$ G-invariant metric on M.
- M/G is compact$\Rightarrow$ compact $Y \subset M$ such that $G(Y) = M$.
- inner product

$$\langle \alpha, \beta \rangle_0 = \int_M \alpha \wedge *\beta.$$

- there exist U, U', two open subsets of M, such that $Y \subset U$ and that the closures $\overline{U}$ and $\overline{U'}$ are both compact in M, and that $\overline{U} \subset U'$.

- G acts on M properly$\Rightarrow$ G-invariant metric on M.
- M/G is compact$\Rightarrow$ compact $Y \subset M$ such that $G(Y) = M$.
- inner product

$$\langle \alpha, \beta \rangle_0 = \int_M \alpha \wedge * \beta.$$

- there exist U, U', two open subsets of M, such that $Y \subset U$ and that the closures $\overline{U}$ and $\overline{U'}$ are both compact in M, and that $\overline{U} \subset U'$.
- smooth function $f : M \to [0, 1]$ such that $f|_U = 1$ and $\mathrm{Supp}(f) \subset U'$.

- $\Gamma(\Omega^*(M))^G$: subspace of G-invariant sections of $\Omega^*(M)$.

- $\Gamma(\Omega^*(M))^G$: subspace of G-invariant sections of $\Omega^*(M)$.
- For an open set W of M, define

$$\|s\|_{W,0}^2 = \int_W \langle s(x), s'(x)\rangle\, dx, \qquad \|s\|_{W,1}^2 = \|s\|_{W,0} + \langle \Delta(s), s\rangle_{W,0}.$$

- $\Gamma(\Omega^*(M))^G$: subspace of G-invariant sections of $\Omega^*(M)$.
- For an open set W of M, define

$$\|s\|^2_{W,0} = \int_W \langle s(x), s'(x)\rangle \, dx, \qquad \|s\|^2_{W,1} = \|s\|_{W,0} + \langle \Delta(s), s\rangle_{W,0}.$$

- For any $s \in \Gamma(\Omega^*(M))^G$,

$$\|s\|_{U,0} \leq \|fs\|_0 \leq \|s\|_{U',0}.$$

- $\Gamma(\Omega^*(M))^G$: subspace of G-invariant sections of $\Omega^*(M)$.
- For an open set W of M, define

$$\|s\|^2_{W,0} = \int_W \langle s(x), s'(x)\rangle\, dx, \qquad \|s\|^2_{W,1} = \|s\|_{W,0} + \langle \Delta(s), s\rangle_{W,0}.$$

- For any $s \in \Gamma(\Omega^*(M))^G$,

$$\|s\|_{U,0} \leq \|fs\|_0 \leq \|s\|_{U',0}.$$

- U' is covered by a finite number of copies of U.

- $\Gamma(\Omega^*(M))^G$: subspace of G-invariant sections of $\Omega^*(M)$.
- For an open set W of M, define

$$\|s\|_{W,0}^2 = \int_W \langle s(x), s'(x) \rangle \, dx, \qquad \|s\|_{W,1}^2 = \|s\|_{W,0} + \langle \Delta(s), s \rangle_{W,0}.$$

- For any $s \in \Gamma(\Omega^*(M))^G$,

$$\|s\|_{U,0} \leq \|fs\|_0 \leq \|s\|_{U',0}.$$

- U' is covered by a finite number of copies of U.
- If s is a G-invariant section of E, there exists a positive constant $C > 0$ such that for any $s \in \Gamma(\Omega^*(M))^G$,

$$\|s\|_{U',0} \leq C\|s\|_{U,0}.$$

- Let $dg := dm(g)$ be the right invariant Haar measure on G. Define $\chi : G \to \mathbb{R}^+$ by $dm(g^{-1}) = \chi(g)dm(g)$. We define $\mathbf{H}_f^0(M, \Omega^*(M))^G$ to be the completion of the space $\{fs : s \in \Gamma(\Omega^*(M))^G\}$ under the norm $\| \cdot \|_0$ associated to the above inner product.

- Let $dg := dm(g)$ be the right invariant Haar measure on G. Define $\chi : G \to \mathbb{R}^+$ by $dm(g^{-1}) = \chi(g)dm(g)$. We define $\mathbf{H}^0_f(M, \Omega^*(M))^G$ to be the completion of the space $\{fs : s \in \Gamma(\Omega^*(M))^G\}$ under the norm $\|\cdot\|_0$ associated to the above inner product.
- Define $\mathbf{H}^1_f(M, \Omega^*(M))^G$ to be the completion of $\{fs : s \in \Gamma(\Omega^*(M))^G\}$ under a (fixed) first Sobolev norm associated to the inner product. And define $\mathbf{H}^2_f(M, \Omega^*(M))^G$ (and $\mathbf{H}^{-1}_f(M, \Omega^*(M))^G$) to be the completion of the space $\{fs : s \in \Gamma(\Omega^*(M))^G\}$ under the corresponding $\mathbf{H}^2_f$ (and $\mathbf{H}^{-1}_f$ norm).

Proposition (Tang-Y-Zhang)

For any $\mu \in L^2(M, \Omega^*(M))$*, its orthogonal projection onto* $\mathbf{H}^0_f(M, \Omega^*(M))^G$ *can be written as*

$$(P_f\mu)(x) = \frac{f(x)}{(A(x))^2} \int_G \chi(g) f(gx)\mu(gx)\, dg,$$

where

$$A(x) = \left(\int_G \chi(g)(f(gx))^2\, dg \right)^{1/2}$$

is a G-equivariant function on M, i.e. $A(gx)^2 = \chi(g)^{-1} A(x)^2$*, and is strictly positive.*

$$\begin{array}{rcl} d_f : \mathbf{H}_f^0 & \to & \mathbf{H}_f^0 \\ f\alpha & \mapsto & f\,d\alpha. \end{array}$$

adjoint $d_f^* : \mathbf{H}_f^0 \to \mathbf{H}_f^0$: $\langle d_f(f\alpha), f\beta\rangle_0 = \langle f\alpha, d_f^*(f\beta)\rangle_0$.

$$\begin{array}{rcl} d_f : \mathbf{H}^0_f & \to & \mathbf{H}^0_f \\ f\alpha & \mapsto & f\, d\alpha. \end{array}$$

adjoint $d_f^* : \mathbf{H}^0_f \to \mathbf{H}^0_f$: $\langle d_f(f\alpha), f\beta\rangle_0 = \langle f\alpha, d_f^*(f\beta)\rangle_0$.
For $f \in C^\infty(M)$, denote $\nabla(f)$ to be the gradient vector field associated to the riemannian metric on M. We can show easily that

$$d_f^*(f\beta) = P_f\left(-\frac{1}{f} i_{2f\nabla f}\beta + f\delta\beta\right) = P_f(-2i_{\nabla f}\beta) + f\delta\beta,$$

where $\delta = *^{-1}d*$ and $i_V\alpha$ is the contraction of the form α with the vector field V.

$$\begin{array}{rcl} d_f : \mathbf{H}_f^0 & \to & \mathbf{H}_f^0 \\ f\alpha & \mapsto & f\, d\alpha. \end{array}$$

adjoint $d_f^* : \mathbf{H}_f^0 \to \mathbf{H}_f^0$: $\langle d_f(f\alpha), f\beta\rangle_0 = \langle f\alpha, d_f^*(f\beta)\rangle_0$.
For $f \in C^\infty(M)$, denote $\nabla(f)$ to be the gradient vector field associated to the riemannian metric on M. We can show easily that

$$d_f^*(f\beta) = P_f\left(-\frac{1}{f} i_{2f\nabla f}\beta + f\delta\beta\right) = P_f(-2i_{\nabla f}\beta) + f\delta\beta,$$

where $\delta = *^{-1}d*$ and $i_V\alpha$ is the contraction of the form α with the vector field V. Now we define a self-adjoint operator

$$\tilde{\Delta} = d_f d_f^* + d_f^* d_f.$$

Proposition (Tang-Y-Zhang)

$\tilde{\Delta} : \mathbf{H}^2_f \to \mathbf{H}^0_f$ *is Fredholm.*

Proposition (Tang-Y-Zhang)

$\tilde{\Delta} : \mathbf{H}^2_f \to \mathbf{H}^0_f$ is Fredholm.

Corollary

$\dim(\ker \tilde{\Delta}) = \dim(\operatorname{coker} \tilde{\Delta}) < +\infty$.

Proposition (Tang-Y-Zhang)

$\tilde{\Delta} : \mathbf{H}^2_f \to \mathbf{H}^0_f$ *is Fredholm.*

Corollary

$\dim(\ker \tilde{\Delta}) = \dim(\operatorname{coker} \tilde{\Delta}) < +\infty$.

Lemma

$\ker \tilde{\Delta} = (\operatorname{Im} \tilde{\Delta})^{\perp} \cap \mathbf{H}^2_f$.

$$\ker \tilde{\Delta} = (\operatorname{Im} \tilde{\Delta})^{\perp},$$

$$\ker \tilde{\Delta} = (\operatorname{Im} \tilde{\Delta})^{\perp},$$

i.e., we have the decomposition

$$\mathbf{H}_f^0 = \ker \tilde{\Delta} \oplus \operatorname{Im} \tilde{\Delta}.$$

- projection $H : \mathbf{H}_f^0 \to \ker \tilde{\Delta}$.

- projection $H : \mathbf{H}^0_f \to \ker \tilde{\Delta}$.
- $f\alpha \in \mathbf{H}^0_f$, $f\alpha - H(f\alpha) \in \mathrm{Im}\, \tilde{\Delta}$.

- projection $H : \mathbf{H}_f^0 \to \ker \tilde{\Delta}$.
- $f\alpha \in \mathbf{H}_f^0$, $f\alpha - H(f\alpha) \in \operatorname{Im} \tilde{\Delta}$.
- there is a unique $f\beta \in \operatorname{Im} \tilde{\Delta}$ such that

$$\tilde{\Delta}(f\beta) = f\alpha - H(f\alpha).$$

- projection $H : \mathbf{H}^0_f \to \ker \tilde{\Delta}$.
- $f\alpha \in \mathbf{H}^0_f$, $f\alpha - H(f\alpha) \in \operatorname{Im} \tilde{\Delta}$.
- there is a unique $f\beta \in \operatorname{Im} \tilde{\Delta}$ such that

$$\tilde{\Delta}(f\beta) = f\alpha - H(f\alpha).$$

- define the Green operator $G : f\alpha \mapsto f\beta$.

- projection $H : \mathbf{H}_f^0 \to \ker \tilde{\Delta}$.
- $f\alpha \in \mathbf{H}_f^0$, $f\alpha - H(f\alpha) \in \operatorname{Im} \tilde{\Delta}$.
- there is a unique $f\beta \in \operatorname{Im} \tilde{\Delta}$ such that

$$\tilde{\Delta}(f\beta) = f\alpha - H(f\alpha).$$

- define the Green operator $G : f\alpha \mapsto f\beta$.

Proposition

Let $\{f\alpha_n\}$ be a sequence of smooth p-forms in $\mathbf{H}_f^2(M, \Omega^*(M))^G$ such that $\|f\alpha_n\|_0 \leqslant c$ and $\|\tilde{\Delta}(f\alpha_n)\|_0 \leqslant c$ for all n and for some constant $c > 0$. Then it has a Cauchy subsequence.

Proposition

If $f\beta$ *is* $\mathbf{H}_f^k(M, \Omega^*(M))^G$ *and*

$$\tilde{\Delta}(f\alpha) = f\beta$$

on M*, then* $f\alpha$ *belongs to* $\mathbf{H}_f^{k+2}(M, \Omega^*(M))^G$ *for any* $k \geq 0$*. In particular, if* $f\beta$ *is a smooth differential form, so is* $f\alpha$*.*

Proposition

Let $\mathfrak{H}^*(M)^G$ denote the kernel of the operator $\tilde{\Delta}$. The map H induces an isomorphism $H : H^p(\Omega^*(M)^G, d) \to \mathfrak{H}^*(M)^G$.

谢谢!